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THE EVALUATION OF CERTAIN TWO-DIMENSIONAL SINGULAR INTEGRALS USED IN THREE-DIMENSIONAL ELASTICITY

J. G. KAZANTZAKIS and P. S. THEOCARIS
Department of Theoretical and Applied Mechanics, The National Technical University of Athens, 5 K. Zographou Street, Zographou, Athens (625), Greece

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Abstract—A method is presented for the investigation of certain two-dimensional singular integrals, frequently accounted in the three-dimensional theory of elasticity. The involved singularity in these integrals was reduced to a complex singularity[3]. The validity of the method was proved in typical cases of three-dimensional elasticity.

1. INTRODUCTION

A method is presented for the investigation of certain two-dimensional singular integrals which are frequently accounted in the three-dimensional stress analysis when the range of integration is a plane region[1]. First, it may be observed that the principal values of the related singular integrals remain the same, regardless of the shape of two types of infinitesimal surfaces, that is the quadrangular and the circular, which are surrounding the existing poles and which must be excluded in order to define the integrals. Based on this fact, we considered the two-dimensional integral as an iterated one[2] and we succeeded to analyze the involved singularity into a pair of complex poles[3]. Thus, we derived a general relation for the investigation of the related singular integrals. Furthermore, by applying this method we succeeded to evaluate directly and in a closed form certain typical integrals, which were previously investigated by Cruse[1] in a different, less general, way. The observed coincidence of the results indicates the validity of the method. The generality of the method as far as its numerical applications are concerned, is also examined[4].

2. FORMULATION OF THE PROBLEM

Consider the two-dimensional singular integral[5] on a plane finite or infinite region S:

$$I(\xi_0, \eta_0) = \int_S \frac{f(\xi_0, \eta_0, \vartheta)}{r^2} u(\xi, \eta)\, d\xi\, d\eta, \tag{1}$$

where

$$(\xi - \xi_0) + i(\eta - \eta_0) = re^{i\vartheta}. \tag{2}$$

The point $\mathbf{x}(\xi_0, \eta_0)$ is called the pole, the functions $f(\xi_0, \eta_0, \vartheta)$ and $u(\xi, \eta)$ are called the characteristic and the density of the singular integral (1). Under the assumptions that (i) the density $u(\xi, \eta)$ is bounded and Hölder continuous function in S, (ii) if S has points at infinity $u(\xi, \eta) = 0(r^{-k})$ $(k > 0)$, and (iii) the characteristic $f(\xi_0, \eta_0, \vartheta)$ is bounded and for a fixed $\mathbf{x}(\xi_0, \eta_0)$ is continuous with respect to ϑ, Tricomi[6] showed that the necessary and sufficient condition for the existence of the singular integral (1) in the principal value sense is that its characteristic satisfies the condition:

$$\int_0^{2\pi} f(\xi_0, \eta_0, \vartheta)\, d\vartheta = 0. \tag{3}$$

Integrals of the form (1) are frequently encountered in the three-dimensional theory of elasticity[1], where (when the range of integration is plane) regardless of the problem the following form of characteristic $f_{kj}(\vartheta)$ is derived[7]:

$$f_{kj}(\vartheta) = l \begin{Vmatrix} 0 & 0 & -\cos\vartheta \\ 0 & 0 & -\sin\vartheta \\ \cos\vartheta & \sin\vartheta & 0 \end{Vmatrix}, \tag{4}$$

where $l = -[(1-2\nu)/8\pi(1-\nu)]$, ν is Poisson's ratio. It is observed that the function $f_{kj}(\vartheta)$ satisfies the condition (3). Thus the singular integral (1) exists in the principal value sense.

Evidently, the difficulties in evaluating (1) are due to its singularity at $r=0$. Far from the vicinity of this point it generally behaves like a regular integral. Thus (1) is to be evaluated in the Cauchy principal value sense which means that an infinitesimal region surrounding the singular point is to be excluded and the limiting value of the integral is to be considered as that region shrinks toward a zero area. The following theorem is valid.

Theorem: ***The principal value of the integral*** **(1)** ***with a characteristic of the form*** **(4)** ***remains the same, if the shape of the excluded area surrounding the pole* x *is circular or quadrangular.***

In order to prove this theorem, let us consider an arbitrary shaped region σ_ϵ surrounding the pole **x**, supposing that the diameter of this region tends to zero together with ϵ. Then the principal value of (1) is given by[7]:

$$\lim_{\epsilon\to 0}\int_{S-\sigma_\epsilon} \frac{f(\xi_0,\eta_0,\vartheta)}{r^2} u(\xi,\eta)\,d\xi\,d\eta = \int_S \frac{f(\xi_0,\eta_0,\vartheta)}{r^2} u(\xi,\eta)\,d\xi\,d\eta - \\ - u(\xi_0,\eta_0)\int_{-\pi}^{+\pi} f(\xi_0,\eta_0,\vartheta)\ln\beta_{\sigma_\epsilon}(\xi_0,\eta_0,\vartheta)\,d\vartheta, \tag{5}$$

with

$$\lim_{\epsilon\to 0}\frac{\alpha(\epsilon,\xi_0,\eta_0,\vartheta)}{\epsilon} = \beta_{\sigma_\epsilon}(\xi_0,\eta_0,\vartheta) > 0, \tag{6}$$

where $r = \alpha(\epsilon,\xi_0,\eta_0,\vartheta)$ is the equation of the boundary of the region σ_ϵ. It is obvious that the value of the second integral on the right of (5) depends upon the shape of the region σ_ϵ surrounding the pole **x**.

Let us consider the local coordinate system ξ, η, ζ with origin at **x** and the ζ-axis along the normal to the plane S at **x** Fig. 1. If σ_ϵ is considered as the circular region K_ϵ, then eqn (5) may be rewritten as:

$$I_A = \lim_{\epsilon\to 0}\int_{S-K_\epsilon} \frac{f(\xi_0,\eta_0,\vartheta)}{r^2} u(\xi,\eta)\,d\xi\,d\eta \\ = \int_S \frac{f(\xi_0,\eta_0,\vartheta)}{r^2} u(\xi,\eta)\,d\xi\,d\eta. \tag{7}$$

If the region σ_ϵ coincides with the quadrangle T_ϵ of Fig. 1, then the following relation is valid:

$$I_B = \lim_{\epsilon\to 0}\int_{S-T_\epsilon} \frac{f(\xi_0,\eta_0,\vartheta)}{r^2} u(\xi,\eta)\,d\xi\,d\eta = \int_S \frac{f(\xi_0,\eta_0,\vartheta_0)}{r^2} u(\xi,\eta)\,d\xi\,d\eta - \\ - u(\xi_0,\eta_0)\int_{-\pi}^{+\pi} f(\xi_0,\eta_0,\vartheta)\ln\beta_{T_\epsilon}(\xi_0,\eta_0,\vartheta)\,d\vartheta. \tag{8}$$

The singular integral over the area $T_\epsilon - K_\epsilon$ (the shaded area) is evaluated as

$$I_A - I_B = \lim_{\epsilon\to 0}\int_{T_\epsilon-K_\epsilon} \frac{f(\xi_0,\eta_0,\vartheta)}{r^2} u(\xi,\eta)\,d\xi\,d\eta = u(\xi_0,\eta_0) \\ \times\int_{-\pi}^{+\pi} f(\xi_0,\eta_0,\vartheta)\ln\beta_{T_\epsilon}(\xi_0,\eta_0,\vartheta)\,d\vartheta. \tag{9}$$

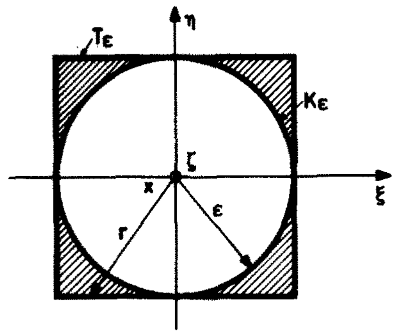

Fig. 1. Geometry of the square and the circle used in the definition of principal value integrals.

Because of the symmetry of the quadrangle and the validity of the following relation for the characteristic (4)

$$f(\xi_0, \eta_0, \vartheta + \pi) = -f(\xi_0, \eta_0, \vartheta), \tag{10}$$

the last integral (9) vanishes. Thus

$$I_A = I_B \tag{11}$$

and the proof is completed.

3. THE PRINCIPAL VALUE OF THE INTEGRAL

If we suppose that the characteristic $f(\xi_0, \eta_0, \vartheta)$ is of the form (4), we can consider the integral (1) as an iterated one[2], analysing the involved singularity into a pair of complex poles, the influence of which can be estimated either in closed form or numerically in [3]. The idea of subtracting a quadrangle instead of a circular disk becomes now useful because with the aid of this idea the limits of integration of the involved definite integrals are now estimated directly. We suppose for simplicity that the region of integration is a square (with side length 2a). Generally it is valid that:

$$\frac{1}{(\xi-\xi_0)^2+(\eta-\eta_0)^2} = \frac{1}{2i}\frac{1}{\xi-\xi_0}\left[\frac{1}{\eta-\eta_0-i(\xi-\xi_0)} - \frac{1}{\eta-\eta_0+i(\xi-\xi_0)}\right]. \tag{12}$$

With the aid of this relation, the principal value of integral (1) may be expressed as

$$I(\xi_0, \eta_0) = \frac{1}{2i}\lim_{\epsilon\to 0}\left[\int_{-\alpha}^{\xi_0-\epsilon}\frac{(I_1-I_2)\,d\xi}{\xi-\xi_0} + \int_{\xi_0+\epsilon}^{\alpha}\frac{(I_1-I_2)\,d\xi}{\xi-\xi_0}\right], \tag{13}$$

where

$$I_1 = \int_{-\alpha}^{\eta_0-\epsilon}\frac{f(\xi_0, \eta_0, \vartheta)u(\xi, \eta)\,d\eta}{\eta - z} + \int_{\eta_0+\epsilon}^{\alpha}\frac{f(\xi_0, \eta_0, \vartheta)u(\xi, \eta)\,d\eta}{\eta - z}, \tag{14}$$

$$I_2 = \int_{-\alpha}^{\eta_0-\epsilon}\frac{f(\xi_0, \eta_0, \vartheta)u(\xi, \eta)\,d\eta}{\eta - \bar{z}} + \int_{\eta_0+\epsilon}^{\alpha}\frac{f(\xi_0, \eta_0, \vartheta)u(\xi, \eta)\,d\eta}{\eta - \bar{z}}, \tag{15}$$

$$z = \eta_0 + i(\xi - \xi_0). \tag{16}$$

The extended application of eqn (13) is discussed completely in a separate paper[4]. The importance of it is based on the fact that the integrals I_1, I_2 can be evaluated numerically[3, 8]. Then, by using the theory of one-dimensional singular integrals[9, 10], a numerical procedure is established for the two-dimensional singular integral (13). In [4] we are proposing the calculation of a set of points on which the numerical integration of a singular integral is performed in the same manner as for a common integral.

4. APPLICATION OF THE SUGGESTED METHOD

There are few examples [1, 11] where an exact determination (in closed form) of the Cauchy principal value of certain two-dimensional singular integrals is given; Cruse [1], using a form of Stokes' theorem succeeds to reduce the surface integral into a line integral (over the contour of the range of integration (which is usually considered as a triangle) and thus he succeeds to evaluate the integral directly. Furthermore, it is obvious that the method of Cruse is not a general one and therefore it can not be considered as a method of evaluating any two-dimensional singular integral. However, this method offers the possibility to compare our general method in the special cases already solved by Cruse. Thus, using Stokes' theorem over the quadrangle of Fig. 2, for the two dimensional singular integral I_3 we obtain:

$$I_3 = \int_S \frac{\partial r/\partial\xi \, d\xi \, d\eta}{r^2} = \ln\left[\frac{(\alpha - \eta_0 + r_C)(-\alpha - \eta_0 + r_A)}{(-\alpha - \eta_0 + r_D)(\alpha - \eta_0 + r_B)}\right]. \tag{17}$$

Observing that the characteristic of I_3 is an element of the matrix (4) and the condition (3) for the existence of the integral is satisfied, we calculate it directly using the suggested method and we find that

$$\begin{aligned} I &= \ln(\alpha - \eta_0 + \sqrt{[(\alpha+\xi_0)^2 + (\alpha-\eta_0)^2]}) \\ &\quad - \ln(-(\alpha+\eta_0) + \sqrt{[(\alpha+\xi_0)^2 + (\alpha+\eta_0)^2]}) \\ &\quad + \ln(-(\alpha+\eta_0) + \sqrt{[(\alpha-\xi_0)^2 + (\alpha+\eta_0)^2]}) \\ &\quad - \ln(\alpha - \eta_0 + \sqrt{[(\alpha-\xi_0)^2 + (\alpha-\eta_0)^2]}) \\ &= \ln\left[\frac{(\alpha - \eta_0 + r_C)(-\alpha - \eta_0 + r_A)}{(-\alpha - \eta_0 + r_D)(\alpha - \eta_0 + r_B)}\right] \end{aligned} \tag{18}$$

Thus, we have derived the same result as in (17). This coincidence of results proves the validity and efficiency of the method. The same results can be obtained by the use of polar coordinates as described in [12]. Furthermore, it may be indicated that the suggested method is a general one since, as was shown in [4], with the aid of this method, the results which are valid for one-dimensional singular integrals [9, 10] are directly generalized for the case of two-dimensional singular integrals. As a consequence, two-dimensional singular integrals of the form:

$$I_4 = \int_S \frac{w(x, y, x_0, y_0)\, dx\, dy}{(x-x_0)^2 + (y-y_0)^2}, \tag{19}$$

which are widely used in three-dimensional elasticity can be investigated by this method for the case where $w(x, y, x_0, y_0)$ is a weight function compatible with the restrictions assuring the existence of the Cauchy principal values of integrals.

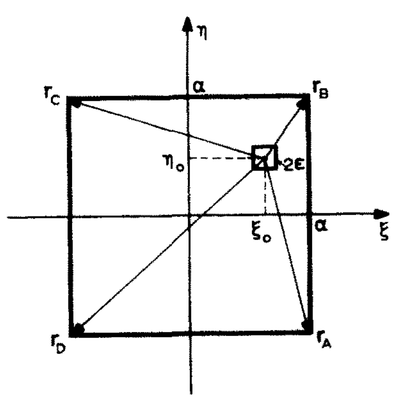

Fig. 2. Geometry of the domain of integration.

REFERENCES

1. T. A. Cruse, Numerical solutions in three dimensional elastostatics. *Int. J. Solids Structures* **5**, 1259 (1969).
2. A. H. Stroud, *Approximate Calculation of Multiple Integrals*, 1st Edn. Prentice-Hall, Englewood Cliffs, New Jersey (1971).
3. F. G. Lether, Subtracting out complex singularities in numerical integration. *Math. Comp.* **31**, 223 (1977).
4. J. G. Kazantzakis and P. S. Theocaris, On the numerical evaluation of two-dimensional singular integrals defined over a rectangle. *J. Elast.* (To be published).
5. S. G. Mikhlin, *Multidimensional Singular Integrals and Integral Equations*, 1st Edn. Pergamon Press, Oxford (1965).
6. F. G. Tricomi, Equazioni integrali contenenti il valor principale di un integrale doppio. *Math. Zeit.* **27**, 87 (1928).
7. V. D. Kupradze, *Potential Methods in the Theory of Elasticity*, 1st Edn. Israel Program for Scientific Translations, Jerusalem (1965).
8. D. B. Hunter, Some Gauss-type formulae for the evaluation of Cauchy principal values of integrals. *Num. Math.* **19**, 419 (1972).
9. P. S. Theocaris, On the numerical solution of Cauchy type singular integral equations. *Serdica, Bulg. Math. Publ.* **2**, 252 (1976).
10. P. S. Theocaris and N. I. Ioakimidis, Application of the Gauss, Radau and Lobatto numerical integration rules to the solution of singular integral equations. *ZAMM* (1978) (To be published).
11. J. Weaver, Three-dimensional crack analysis. *Int. J. Solids Structures* **13**, 321 (1977).
12. P. S. Theocaris, N. I. Ioakimidis and J. G. Kazantzakis, On the numerical evaluation of two-dimensional principal value integrals. *Int. J. Num. Meth. Engng* (To be published).